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Electrodynamics: A Consequence of Nonlinear Realizations of the Lorentz Group[1]

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Extensions from the representations of the Lorentz group to include local nonlinear diagonal transformations is sufficient to generate, via the covariant derivative, the interaction of minimal coupling. These diagonal realizations are characterized by six functions ϕ_i which must satisfy a system of transformation equations. Inequivalent categories of solutions for the ϕ_i give rise to different electromagnetic fields. The Dirac monopole and Coulomb potentials follow directly from two different categories of these nonlinear realizations. Within this theory, charge becomes simply the nonlinear counterpart of intrinsic spin for a *particular* nonlinear realization of the Lorentz group. Charge is thus placed on equal footing with intrinsic spin in the sense that both phenomena can be described as consequences of our space-time symmetry. Other solutions for the six ϕ_i exist, including a spinor. We briefly discuss the possibility that with these other solutions, these realizations could represent some other basic properties of elementary particles.

1. INTRODUCTION

A primary motivation for considering nonlinear realizations of groups is that, like Yang–Mills gauge groups, they are generally local in nature and consequently give rise to interactions introduced through covariant derivatives.[2] Although several varieties of nonlinear realizations of the Poincaré

[1] Presented at the Dirac Symposium, Loyola University, New Orleans, May 1981.
[2] See Weinberg (1968). In addition, some of the other pioneering articles on this subject are Coleman et al. (1969), Callan et al. (1969), and Joseph and Solomon (1970). See also Salam and Strathdee (1969). A recent discussion for noncompact groups has been given by Julia and Luciani (1980).

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group, or its subgroups, have been published (Poincaré, 1896[3]; Hind, 1971, 1972; Hopkinson and Reya, 1972; Dalton 1978; Philips and Wigner, 1968; Melvin, 1962, 1963; Takabayasi, 1966; Dalton, 1979), few significant applications of the latter to the structure, or interactions of elementary particles have been found. Nevertheless, the dominating role in physical applications already played by the space-time symmetries, combined with the fact that nonlinear realizations can generate interactions, is strong motivation for further study on this subject.

Last year (1980), I derived (Dalton, 1980) the classical Lorentz force equations from the covariant acceleration equations associated with nonlinear realizations of the Lorentz group acting on the four-velocities of particles. In a more recent extension (Dalton, to be published) of this work, a set of solutions have been found which are characterized by six functions ϕ_i that must satisfy a set of nonlinear transformation equations. This work is all in the context of classical mechanics.

In the study presented here, I consider nonlinear realizations of the Lorentz group as a transformation group acting on the wave functions ψ and potentials A_μ. These realizations are restricted to a simple diagonal nonlinear extension of conventional representations. These realizations are characterized by six functions ϕ_i that must satisfy the same particular transformation equations arrived at in the classical mechanics study mentioned above. In the quantum mechanical context these six functions have a simple interpretation. They are the diagonal components for the generators representing the transformation. We show that the local nature of these transformations generate, via the covariant derivative, the interaction potential of minimal coupling. Different solutions of the transformation equations give rise to different potentials. For two different solutions we obtain the magnetic monopole and Coulomb potentials. Both solutions are derived without the use of Maxwell's equations.

Within the context of a Lagrangian field theory, we show that the angular momentum generated by the linear and nonlinear parts are separately conserved if the Lagrangian represents a closed system (that is, if we include both ψ and the field $F_{\mu\nu}$ in the Lagrangian). Since there is exact consistency between the observed angular momentum conservation and that described with linear realizations (representations), this key development removes a major obstacle in physical applications of these nonlinear realizations of the Lorentz group.

In Appendix A we outline a derivation for the commutator relation for arbitrary (linear or nonlinear) transformations and discuss the forms ap-

[3]In this work the Lorentz force equations for a magnetic monopole field were integrated to obtain first integral angular momentum expressions that involved, in addition to the usual orbital term, a nonlinear term; representing in fact a nonlinear realization of O(3).

propriate to the study here. A derivation of the covariant derivative, (that is, how the potential must transform) is given in Appendix B. The reader not familiar with nonlinear realizations is advised to first read, or work through, these two appendices. In Section 2 we discuss the diagonal nonlinear realizations and certain features related to the electromagnetic field. The conservation theorems are discussed in Section 3. We describe the equations in a convenient vector basis in Section 4. Section 5 is dedicated to a derivation of the monopole potential from the nonlinear transformation equations for the $SU(2)$ subgroup. The particular realization that gives rise to the Coulomb potential is discussed in Section 6.

The development in this paper is in the context of *classical* quantum mechanics and field theory. However, most of the results will not differ in a quantum field theory development.

2. DIAGONAL REALIZATIONS AND COVARIANT DERIVATIVES

The symmetry group considered here is the six-parameter homogeneous Lorentz group [actually $SL(2,C)$ the twofold covering group] under which a set of coordinates $x \equiv \{x_\mu | \mu = 1-4\}$ of a point in Minkowski space is transformed to a set of coordinates $x' \equiv \{x'_\mu\}$ by the usual linear (four-vector) transformation. The Pauli metric notation is used with $x_4 = ict$ together with the convention of summing over repeated indices. The symbols α, β, γ are used for sets of six group parameters, $\alpha = \{\alpha_i | i = 1-6\}$. We are considering a transformation group on a space of N-dimensional complex functions $\psi, \psi^T = (\psi_1, \psi_2, \ldots \psi_N)$ (i.e., the wave function or field operator). The particular infinitesimal transformations on ψ considered here are restricted to a linear part, which generates the intrinsic spin, plus a diagonal nonlinear part; the combination is indicated as follows:

$$\psi'(x') = \psi(x) + \alpha_i(\phi_i 1 - t_i)\psi(x) \tag{1}$$

where 1 is the $N \times N$ unit matrix and the t_i generate an $N \times N$ matrix representation. The six functions ϕ_i, $i = 1,6$, introduced in (1) are not inert under the group action and are in general space-time dependent so that $\partial_\mu \phi_i \neq 0$ where $\partial_\mu = \partial / \partial x_\mu$. From Appendix A we see that the commutator relations for the above transformation of ψ reduce to the following equations:

$$(\delta_i \phi_j) - (\delta_j \phi_i) = C_{ijk} \phi_k \tag{2}$$

$$[t_i, t_j] \equiv t_i t_j - t_j t_i = C_{ijk} t_k \tag{3}$$

Here, the C_{ijk} are the group structure constants and $(\delta_i\phi_j)$ is defined (see Appendix A) in the infinitesimal transformation of ϕ_j, that is,

$$\phi_j'(x') = \phi_j(x) + \alpha_i(\delta_i\phi_j) \tag{4}$$

In addition to (2) and (3) which represent commutator relations for the transformation on ψ, the following commutator relations for the transformation on the ϕ_i must also be satisfied:

$$\big(\delta_i(\delta_j\phi_k)\big) - \big(\delta_j(\delta_i\phi_k)\big) = C_{ije}(\delta_e\phi_k) \tag{5}$$

These equations (5), may, or may not, represent extra conditions [that is, in addition to (2)]. For instance, if the terms $(\delta_i\phi_k)$ and $(\delta_i(\delta_j\phi_k))$ are functions of c-number variables which themselves satisfy the commutator relations, then (5) will automatically be satisfied. On the other hand, if the ϕ_k themselves are taken to be the basic variables, then the equations in (5) must be solved simultaneously with (2). We point out that solutions of (5) can be found which will not satisfy (2). For instance, suppose we consider the solution $(\delta_i\phi_j) = C_{ijk}\phi_k$ for a self-representation. If we use this solution of (5) in (2) together with $C_{ijk} = -C_{jik}$, we obtain $2 = 1$, a contradiction. On the other hand, consider an arbitrary symmetric function S_{ij} ($S_{ij} = S_{ji}$). The form $(\delta_i\phi_j) = S_{ij} + \frac{1}{2}C_{ijk}\phi_k$ will satisfy (2) but the equations in (5) can be solved for only certain choices of functions S_{ij}. It should be noted here that all of the nonlinear realizations studied in Dalton (1979) involved the antisymmetric term $\frac{1}{2}C_{ijk}\phi_k$. Considering all of the indices in (2) and (5) these expressions represent about 30 transformation equations which the six ϕ_i must satisfy. This large number of equations is rather prohibitive. However, a variety of solutions exist, ranging from spinors to particular nonlinear realizations.[4] Detailed solutions for some cases will be given in Sections 4–6, and the reader should see Dalton (1979) for others.

Since $\partial_\mu\phi_i \neq 0$, expressions such as $\psi\gamma_\mu\partial_\mu\psi$ will not be invariant. Construction of invariant forms involving $\partial_\mu\psi$ is facilitated by first constructing a covariant derivative $D_\mu\psi$ which transforms like ψ (apart from the index μ if we are considering a space-time group). For this purpose we consider the following form:

$$D_\mu\psi = (\partial_\mu + A_\mu)\psi \tag{6}$$

Notice that for simplicity, we have absorbed the usual factor of $i = \sqrt{-1}$

[4]By identifying the six functions ϕ_i with the real and imaginary parts of the complex functions S_i of Dalton (1979) one can take advantage of this previous development to obtain several solutions of the thirty transformation equations.

and coupling constant into A_μ. From Appendix B we have the following transformation rule for the potential A_μ:

$$A'_\mu = A_\mu - \alpha_i s_i^{\mu\rho} A_\rho - \alpha_i \partial_\mu \phi_i \tag{7}$$

Here, the quantities $s_i^{\mu\rho}$ are the elements of the four-vector representation generators of the Lorentz group. With our notation we have

$$\alpha_i(\delta_i A_\mu) \equiv A'_\mu - A_\mu \tag{8}$$

which with (7) leads to

$$(\delta_i A_\mu) = -s_i^{\mu\rho} A_\rho - \partial_\mu \phi_i \tag{9}$$

Factoring out the parameters to obtain (9) is possible since $\partial_\mu \alpha_i = 0$. This, of course, would not be possible if we were also considering a group of local gauge transformations since in that case $\partial_\mu \alpha_i \neq 0$. Equation (9) is also valid for an internal group if we drop the first term on the right-hand side. On inspection of (9), we see that A_μ does *not* transform as a four-vector unless $\partial_\mu \phi_i = 0$.

In the above we have started with the transformation on ψ and arrived at the transformation on the A_μ. We could have started with (9) as a more general Lorentz transformation on the A_μ that does not change the usual transformation of the $F_{\mu\nu}$ (except at discontinuities in the ϕ_i). Imposing the commutator relations in the action on the A_μ leads directly to equations (2) so that the latter equations are not restricted to quantum mechanics. A second point is that if we were considering a quantum field theory,[5] we would replace $\phi_i \psi$ in (A.20) with $(\phi_i \psi + \psi \phi_i)/2$.

Even though A_μ in (7) does not transform as a four-vector, this theory is consistent (within the exceptions discussed below) with the conventional Maxwell theory expressed in terms of field tensors $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. From (7) it is easy to show that the $F_{\mu\rho}$ have the following transformation properties:

$$F'_{\mu\rho} = F_{\mu\rho} - \alpha_i\left[s_i^{\mu\alpha} F_{\alpha\rho} - s_i^{\rho\alpha} F_{\alpha\mu}\right] - (\partial_\mu \partial_\rho - \partial_\rho \partial_\mu)\alpha_i \phi_i \tag{10}$$

The first two terms on the right-hand side of (10) correspond to the usual infinitesimal Lorentz transformation on $F_{\mu\nu}$. The last term vanishes if ϕ_i is an integrable function of space-time. If we *define* the current components j_μ as follows:

$$j_\mu \equiv \partial_\nu F_{\nu\mu} \tag{11}$$

[5]See for instance Hammer and Good (1961).

then we have the following transformation properties:

$$j'_\mu = j_\mu - \alpha_i s_i^{\mu\rho} j_\rho - \partial_\rho(\partial_\rho\partial_\mu - \partial_\mu\partial_\rho)\alpha_i\phi_i \tag{12}$$

From this we see that the components j_μ transform as four-vectors if the ϕ_i are integrable functions.

There are two points of interest here. First, if the ϕ_i are not continuous (which we will show happens on the singularity string for the monopole and at $r = 0$ for a point charge), then the usual transformations of $F_{\mu\nu}$ and j_μ are modified. This means for instance that the expression $F_{\mu\nu}F_{\mu\nu}$ will not be invariant under a Lorentz transformation at these discontinuities. The second point of interest is that even where the ϕ_i are continuous, this theory is more restrictive than the conventional Maxwell theory. This arises from the presence in (9) of the $\partial_\mu\phi_i$ term. Since the six ϕ_i are limited to only those solutions which satisfy the 30 equations indicated in (2) and (5) the type of potentials which can satisfy (9) are limited and intrinsically related to the solutions of (2) and (5). As we will demonstrate in the following sections, finding a solution (i.e., explicit functions for the six ϕ_i and four A_μ) gives the same results obtained by solving Maxwell's equations for a particular source. In other words, a solution of (2), (5), and (9) specifies the source of Maxwell's equations. The important question which arises from this feature is "Can a solution of (2), (5), and (9) be found which corresponds to the field of a charged particle?" We will show in Section 6 that the answer to this is yes!

We remark here that the introduction of potentials which transform as in (9) are not new, having been introduced in earlier theories (Hammer and Good, 1961) of quantum electrodynamics as means of making gauge conditions, such as $\partial_i A_i = 0$, Lorentz covariant. The important point to realize, however, is that the local action of the above Lorentz transformation on ψ necessitates the interaction of minimal coupling, and as we shall demonstrate in the following sections, several solutions of (2), (5), and (9) exist in a given gauge, so that different solutions do not correspond merely to choosing different gauges. It should be remarked, however, that certain solutions may exist that are particular to a given gauge.

3. CONSERVATION RULES

Here we discuss the conservation of probability and angular momentum associated with the extended Lorentz transformations described in the previous section. For the conservation of probability let $\bar{\psi}\psi$ ($\bar{\psi} = \psi^+\Gamma$) represent the probability density that is invariant for a representation of

$SL(2, C)$ generated by the t_i. The matrix Γ must be appropriately chosen for the representations considered. For the extended transformations of the previous section, we have the following relation:

$$\begin{aligned}(\bar{\psi}\psi)' &= \bar{\psi}\psi + (1+\alpha_i\phi_i^\star)(1+\alpha_i\phi_i) \\ &= \bar{\psi}\psi + \alpha_i(\phi_i^\star + \phi_i)\bar{\psi}\psi\end{aligned} \tag{13}$$

Recall that in our notation we have absorbed the usual factor of $\sqrt{-1}$ in the ϕ_i and t_i, and have chosen real parameters. From (13) we obtain the expected result that $\bar{\psi}\psi$ is invariant if ϕ_i is pure imaginary. In other words, we have invariance of the probability density if the diagonal part of the transformation is a phase transformation. Solutions for ϕ_i which have a real component for one or more of the six ϕ_i functions would not be acceptable in our present picture of quantum mechanics. We stress that the sure imaginary property of the ϕ_i must be invariant under the transformations if the latter are to be physical.

To discuss the conserved angular momentum, we consider a Lagrangian density of the form

$$\mathfrak{L} = \left(\Psi_i, (\partial_\mu + A_\mu)\Psi, \partial_\mu A_\rho - \partial_\rho A_\mu\right) \tag{14}$$

As indicated in this expression, we consider only $\mathfrak{L}$'s for which $\partial_\mu \Psi$ and A_μ appear only in the form $(\partial_\mu + A_\mu)\Psi$. This condition leads to the following relation:

$$\frac{\partial \mathfrak{L}}{\partial(\partial_\mu \psi_i)}\psi_i = \frac{\partial \mathfrak{L}}{\partial A_\mu} \tag{15}$$

Likewise, we consider $\mathfrak{L}$'s for which $\partial_\mu A_\rho$ appears only in the form $\partial_\mu A_\rho - \partial_\rho A_\mu$. This constraint may be expressed in the following way:

$$\frac{\partial \mathfrak{L}}{\partial(\partial_\mu A_\nu)} = -\frac{\partial \mathfrak{L}}{\partial(\partial_\nu A_\mu)} \tag{16}$$

Equations (15) and (16) are satisfied for the standard Lagrangians of quantum electrodynamics.

We consider variations of the Lagrangian for which the changes of Ψ_j and A_ρ have the following forms:

$$\delta\Psi_j = \varepsilon_i(\delta_i \Psi_j) = \varepsilon_i\left(\phi_i \Psi_j - t_i^{jk}\Psi_k\right) \tag{17}$$

$$\delta A_\rho = \varepsilon_i(\delta_i A_\rho) = \varepsilon_i\left(-\partial_\rho \phi_i - s_i^{\rho\nu} A_\nu\right) \tag{18}$$

where the ε_i are infinitesimal parameters. With the action

$$W = \int dx^4 \mathfrak{L} \tag{19}$$

and (17), (18), one can easily arrive at the following form:

$$\delta W = \varepsilon_i \int dx^4 \partial_\mu \left\{ \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \psi)} \left[(\delta_i \Psi_j) - \frac{1}{2} s_i^{\sigma\rho} l_{\sigma\rho} \Psi_j \right] + \frac{\partial \mathfrak{L}}{\partial(\partial_\mu A_\alpha)} \left[(\delta_i A_\alpha) - \frac{1}{2} s_i^{\sigma\rho} l_{\sigma\rho} A_\alpha \right] - \mathfrak{L} s_i^{\mu\rho} x_\rho \right\} \tag{20}$$

where $l_{\sigma\rho} \equiv x_\sigma \partial_\rho - x_\rho \partial_\sigma$. Using the expressions for $(\delta_i \psi_j)$ and $(\delta_i A_\rho)$ given above, we can write (20) in the following form:

$$\begin{aligned}\delta W = \varepsilon_i \int dx^4 \Bigg\{ \partial_\mu \Bigg[& \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \Psi_j)} \left(- t_i^{jk} \Psi_k - \frac{1}{2} s_i^{\sigma\rho} l_{\sigma\rho} \Psi_j \right) \\ & + \frac{\partial \mathfrak{L}}{\partial(\partial_\mu A_\alpha)} \left(- s_i^{\alpha\nu} A_\nu - \frac{1}{2} s_i^{\sigma\rho} l_{\sigma\rho} A_\alpha \right) - \mathfrak{L} s_i^{\mu\rho} x_\rho \Bigg] \\ & + \partial_\mu \left[\frac{\partial \mathfrak{L}}{\partial(\partial_\mu \Psi_j)} (\phi_i \psi_j) + \frac{\partial \mathfrak{L}}{\partial(\partial_\mu A_\rho)} (- \partial_\rho \phi_i) \right] \Bigg\}\end{aligned} \tag{21}$$

In this expression we have collected the usual covariant angular momentum tensor density components in the first square bracket ($\equiv M_i^\mu$) and the corresponding contribution from the nonlinear part in the second square bracket ($\equiv N_i^\mu$). If W is invariant under the group of transformations we have from Noether's theorem

$$\partial_\mu (M_i^\mu + N_i^\mu) = 0 \tag{22}$$

This expression simply means that the total covariant angular momentum tensor densities $M_i^\mu + N_i^\mu$ are conserved.

From (21) consider the density N_i^μ:

$$\begin{aligned} N_i^\mu &= \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \psi_j)} \psi_j \phi_i - \frac{\partial \mathfrak{L}}{\partial(\partial_\mu A_\rho)} \partial_\rho \phi_i \\ &= \frac{\partial \mathfrak{L}}{\partial A_\mu} \phi_i - \frac{\partial \mathfrak{L}}{\partial(\partial_\mu A_\rho)} \partial_\rho \phi_i \\ &= \partial_\rho \left(\frac{\partial \mathfrak{L}}{\partial(\partial_\rho A_\mu)} \right) \phi_i + \frac{\partial \mathfrak{L}}{\partial(\partial_\rho A_\mu)} \partial_\rho \phi_i \\ &= \partial_\rho \left[\frac{\partial \mathfrak{L}}{\partial(\partial_\rho A_\mu)} \phi_i \right] \end{aligned} \tag{23}$$

In rearranging terms in the above expression we have used (15), (16), and the Euler–Lagrange equations. For continuous functions ϕ_i we have from (23) and (16) the following conservation equation:

$$\partial_\mu N_i^\mu = \partial_\mu \partial_\rho \left| \frac{\partial \mathfrak{L}}{\partial(\partial_\rho A_\mu)} \phi_i \right| = 0 \tag{24}$$

With this expression it follows from (22) that $\partial_\mu M_i^\mu = 0$. This means that the usual covariant angular momentum tensor density M_i^μ is conserved.

From the same variation we have obtained two separately conserved quantities. However, this *does not* mean that we have two separate Lorentz symmetry groups. The existence in this case of two conserved densities is a simple consequence of the fact that in the variation of the Lagrangian, the net expression involving the usual linear components and that involving the nonlinear components separately add to zero. This result is a direct consequence of our above-mentioned restrictions on $\mathfrak{L}$ and does not reflect anything about the group symmetry. However, we do obtain an additional conserved quantity.

With the extensive role played by the conservation condition $\partial_\mu M_i^\mu = 0$ in particle interactions, the above development raises the possibility that the conservation condition $\partial_\mu N_i^\mu = 0$ may eventually prove likewise useful.

Here we should contrast the above conservation rules with that considered by Fierz (1944; see also Wentzel, 1943) for the charge monopole system. In this previous work the above conservation rule (22) is obtained. However, the conservation conditions $\partial_\mu M_i^\mu = 0$ and $\partial_\mu N_i^\mu = 0$ are *not* obtained. This has led to statements such as "the monopole produces a noncentral force." In this previous publication, and several other since,[6] one finds that the N_i^μ term does not include the second part

$$\frac{\partial \mathfrak{L}}{\partial(\partial_\mu A_\rho)} \partial_\rho \phi_i$$

of (23). This simply means that the Lagrangian used had no dependence on the field tensors $F_{\mu\nu}$. For the standard Lagrangian forms in QED this corresponds to leaving out the $F_{\mu\nu}F_{\mu\nu}$ term. In contrast, the conservation rules discussed above depend on including in $\mathfrak{L}$ all fields that are dynamically involved in the system. One might argue that the contribution to the angular momentum arising from including in $\mathfrak{L}$ the $F_{\mu\nu}$ dependence is small and this can be ignored. This argument is not valid, however, because the contribution to the continuity equation $\partial_\mu N_i = 0$ of this term is exactly equal in magnitude but opposite in sign to that of the first term in N_i^μ. In summary, the previously derived angular momentum conservation rules for the charge monopole system were derived from an incomplete Lagrangian in which the electromagnetic field of the monopole was considered external. The conservation rules given above are derived from a Lagrangian representing a complete, or closed system. It is for such systems that conservation of angular momentum has been an important analytic tool. It is by now traditional (Wu and Yang, 1976; Yang, 1977; Boulware et al., 1976) to diagonalize the J_Z for the incomplete Lagrangian and arrive at quantization rules involving the charge and monopole strengths. However, since the operator involved is not a conserved operator for the complete system one should question the physical validity of these results. As a final remark, we point out that the above conservation theorems can be derived in the framework of classical mechanics if the Lagrangian includes (as it usually does) terms that depend on the electromagnetic field tensors.

4. VECTOR BASIS

To describe specific solutions we find it convenient to use the vector basis of Dalton (1979) and to use $\phi_i(\delta_i)$ for the rotation subgroup and $\hat{\phi}_i(\hat{\delta}_i)$ for the pure Lorentz transformation. With this notation the indices range from 1 to 3. With the structure constants of Dalton (1979) expression (2)

[6] See Wu and Yang (1976) and Yang (1977). A more complete list of references can be found in Boulware et al. (1976).

gives the following equations:

$$(\delta_i \phi_j) - (\delta_j \phi_i) = -\varepsilon_{ijk} \phi_k \tag{25}$$

$$(\delta_i \hat{\phi}_j) - (\hat{\delta}_j \phi_i) = -\varepsilon_{ijk} \hat{\phi}_k \tag{26}$$

$$(\hat{\delta}_i \hat{\phi}_j) - (\hat{\delta}_j \hat{\phi}_i) = +\varepsilon_{ijk} \phi_k \tag{27}$$

The expression in (9) gives the following equations:

$$(\hat{\delta}_i A_4) = iA_i - \partial_4 \hat{\phi}_i \tag{28}$$

$$(\hat{\delta}_i A_j) = -iA_4 \delta_{ij} - \partial_j \hat{\phi}_i \tag{29}$$

$$(\delta_i A_j) = -\varepsilon_{ijk} A_k - \partial_j \phi_i \tag{30}$$

$$(\delta_i A_4) = -\partial_4 \phi_i \tag{31}$$

We have from (5) the relations

$$(\delta_i(\delta_j Z)) - (\delta_j(\delta_i Z)) = -\varepsilon_{ijk}(\delta_k Z) \tag{32}$$

$$(\delta_i(\hat{\delta}_j Z)) - (\hat{\delta}_j(\delta_i Z)) = -\varepsilon_{ijk}(\hat{\delta}_k Z) \tag{33}$$

$$(\hat{\delta}_i(\hat{\delta}_j Z)) - (\hat{\delta}_j(\hat{\delta}_i Z)) = +\varepsilon_{ijk}(\delta_k Z) \tag{34}$$

where Z represents any one of the ϕ_i or $\hat{\phi}_i$. With this basis, the infinitesimal transformation on Z is expressed by

$$Z' = Z + \omega_i(\delta_i Z) + \nu_i(\hat{\delta}_i Z) \tag{35}$$

where both $\boldsymbol{\omega}$ and $\boldsymbol{\nu}$ are real. The reader should notice that the usual factor $i = \sqrt{-1}$ has been absorbed into the $(\delta_i Z)$ terms (see Appendix A for some discussion of this degree of freedom in choosing a basis). If the more traditional basis is used (see footnote 12 of Dalton, 1979), the structure constants above will be modified by a factor of $\sqrt{-1}$.

5. MONOPOLE POTENTIAL FROM $SU(2)$

One of the first published examples of a diagonal nonlinear realization was given in context of a charge-monopole system (Poincaré, 1896). In 1896

Poincaré, in an attempt to explain an experiment of Birkeland (1896), considered the Lorentz force equations with the magnetic field of a monopole. From the first integrals of these equations he obtained an angular momentum, which contained in addition to the usual orbital form, a nonconstant term. After the (1931) quantum mechanical study of the monopole by Dirac (1931, 1948) Fierz (1944) started with the classical force equations with the angular momentum operators discussed by Poincaré (no reference to Poincaré was given) and then considered the quantum mechanical version of these operators. The nonconstant term generated a diagonal nonlinear transformation of $O(3)$ on the wave function. Fierz showed that by diagonalizing the angular momentum (J_z), he could obtain a charge-monopole quantization condition. This approach to charge quantization has since been repeated by many authors (Wu and Yang, 1976; Yang, 1977; Boulware et al., 1976).

In this section we show that the magnetic monopole potential is a direct solution of the above nonlinear realizations for the $SU(2)$ or $O(3)$ subgroup. We also discuss the conserved angular momentum generators for a closed versus open system and the implications for the charge quantization problem.

To solve for ϕ_i we consider the following form:

$$\phi_i = a\varepsilon_{ijk}x_j A_k + F_i \tag{36}$$

where a, A_k, and F_i are to be determined. In our notation we have $(\delta_i x_j) = -\varepsilon_{ijk}x_k$. If we use (36) in (25), we will obtain expressions involving the quantities $(\delta_i A_j)$. For these, we use the expression (30). This introduces the derivatives $\partial_j\phi_i$ which with (36) introduces the derivatives $\partial_i A_j$ into the expression. The net result is the following set of differential equations involving the x_i, A_j, F_i, and a:

$$\begin{aligned}(a+a^2)(x_jA_i - x_iA_j) + \varepsilon_{ije}F_e - (\varepsilon_{ije}x_e)\varepsilon_{k\alpha n}x_\alpha\left[a^2\partial_k A_n - a(\partial_k a)A_n\right]& \\ + a\left[\varepsilon_{iek}x_e\partial_k F_j - \varepsilon_{jek}x_e\partial_k F_i\right] + (\delta_i F_j) - (\delta_j F_i) = 0&\end{aligned} \tag{37}$$

We have used the following identity to reduce to the above expression:

$$(\varepsilon_{jek}\varepsilon_{i\alpha n} - \varepsilon_{iek}\varepsilon_{j\alpha n})x_e x_\alpha = -(\varepsilon_{jie}x_e)(\varepsilon_{k\alpha n}x_\alpha) \tag{38}$$

We consider the solution of (37) for which $a=-1$, and $F_e = Cx_e$, where C is a function of $x=(x_ix_i)^{1/2}$ so that $(\delta_i C)=0$. In this case (37) reduces to the following expression:

$$\varepsilon_{k\alpha n}x_\alpha\partial_k A_n = C \tag{39}$$

To solve (39) we consider the following form for A_n:

$$A_n = g(\mathbf{x} \cdot \mathbf{n}, x^2)\varepsilon_{nej}x_e n_j \tag{40}$$

where n_j is an arbitrary unit vector with $\partial_j n_k = 0$. Using (40) in (39) gives the following differential equation for g:

$$g'\left[(\mathbf{x}\cdot\mathbf{n})^2 - x^2\right] + 2g(\mathbf{x}\cdot\mathbf{n}) = C(x) \tag{41}$$

where $g' = \partial g/\partial(\mathbf{x}\cdot\mathbf{n})$. This equation has the following solutions:

$$g^{\pm} = \frac{C(r)}{[(\mathbf{x}\cdot\mathbf{n}) \pm x]} \tag{42}$$

where $C(x)$ is still arbitrary. To determine $C(x)$, we notice from (40) that $\mathbf{A}\cdot\mathbf{x} = 0$. If the form in (40) is to be covariant we must also have $\mathbf{x}'\cdot\mathbf{A}' = 0$. Imposing this covariant condition gives

$$\mathbf{x}'\cdot\mathbf{A}' = 0 = \mathbf{x}\cdot\mathbf{A} + A_i(\delta x_i) + (\delta A_i)x_i \tag{43}$$

Using $(\delta A) \equiv A_i' - A_i = \alpha_j(\delta_j A_i)$, and $(\delta x_i) = \alpha_j(\delta_j x_i)$ we obtain

$$x_i \partial_i \phi_j = 0 \tag{44}$$

From these equations and a little arithmetic we arrive at the following expression:

$$C'x + C = 0 \tag{45}$$

where $C' \equiv dC/dx$. This equation has the solution $C = \mu/x$, where μ is a constant. Combining the above results we have the following solutions:

$$A_i^{\pm} = \frac{\mu\varepsilon_{ijk}x_j n_k}{x[(\mathbf{x}\cdot\mathbf{n}) \pm x]} \tag{46}$$

$$\phi_i^{\pm} = -\varepsilon_{ijk}x_j A_k^{\pm} + \frac{\mu x_i}{x} \tag{47}$$

The vector potentials given in (46) have the form of the Dirac magnetic monopole potential. In comparison with the work by Dirac we see that the unit vector $\mathbf{n}$ lies along the Dirac string. If $\mathbf{x}$ is parallel to $\mathbf{n}$, one or the other of the above solutions is singular. The linear combinations $A_i^+ \pm A_i^-$ of (47) are also solutions. Both of these solutions have been used in previous applications.[7]

The important difference between the above derivation and previous work is that here solutions are obtained by solving equations (25) and (30) and *not* Maxwell's equations. In summary, the diagonal nonlinear realizations of the rotation group generates (as one solution) the magnetic monopole field.

In the above solution let us consider the nonlinear conserved angular momentum density N_i^μ discussed in Section 3. We have

$$N_i^\mu = \frac{\partial \mathfrak{L}}{\partial(\partial_\mu A_j)} \psi_j \left[-\varepsilon_{iek} x_e A_k + \frac{\mu x_i}{x} \right] - \frac{\partial \mathfrak{L}}{\partial(\partial_\mu A_\rho)} \partial_\rho \left[-\varepsilon_{iek} x_e A_k + \frac{\mu x_i}{x} \right] \tag{48}$$

The second term in this expression is what is left out when one uses a Lagrangian that represents only part of the system. To make this more clear, consider the usual Lagrangian density of QED:

$$\mathfrak{L} = \bar{\Psi} \gamma_\mu (\partial_\mu + A_\mu) \psi - \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} \tag{49}$$

We obtain the result

$$N_i^\mu = \bar{\psi} \gamma_\mu \psi \left[-\varepsilon_{iek} x_e A_k + \frac{\mu x_i}{x} \right] + F_{\mu\rho} \partial_\rho \left[-\varepsilon_{iek} x_e A_k + \frac{\mu x_i}{x} \right] \tag{50}$$

The second expression in (50) would not be included if the $F_{\mu\nu}F_{\mu\nu}$ term was left out of the Lagrangian. If one chooses to approach charge quantization by diagonalizing an angular momentum operator, it would seem logical to choose the above conserved operator N_i^μ.

6. COULOMB POTENTIAL AND THE CHARGE REALIZATION

We now consider a solution of (25)–(34) that gives rise to the Coulomb potential for a point charge. We emphasize that several of the solutions in

[7]See Barut and Bornzin (1971). See also Schwinger (1966). Some problems with this article have been clarified in Wentzel (1966).

Dalton (1979) satisfy the equations of Section 4, but these solutions do not generate the Coulomb potential of a point charge. For this solution we set $\phi_i = 0$ for the angular momentum generators. This means that the $\hat{\phi}_i$ describing the nonlinear pure Lorentz part must satisfy

$$(\hat{\delta}_i \hat{\phi}_j) - (\hat{\delta}_j \hat{\phi}_i) = 0 \tag{51}$$

$$(\delta_i \hat{\phi}_j) = -\varepsilon_{ijk} \hat{\phi}_k \tag{52}$$

From (51) we see that the antisymmetric part of $(\hat{\delta}_i \hat{\phi}_j)$ must be zero. Limiting the $(\hat{\delta}_i \hat{\phi}_j)$ to functions of the $\hat{\phi}_i$ only, we can express them in the following way:

$$(\hat{\delta}_i \hat{\phi}_j) = F \delta_{ij} + h \hat{\phi}_i \hat{\phi}_j \tag{53}$$

where F and h are functions to be determined. With this expression, equations (25)–(27) are satisfied. There are several different solutions of F and h (i.e., different realizations) which will solve (32)–(34). Here, we discuss a particular solution in which the function $\hat{\phi}^2 = \hat{\phi}_i \hat{\phi}_i$ is left invariant. Restricting F and h to be functions only of this invariant quantity we can use (53) in (34) to obtain the following conditions:

$$F = \pm \hat{\phi}, \qquad h = \mp 1/\hat{\phi} \tag{54}$$

where $\hat{\phi}$ is one of the roots of $\hat{\phi}^2$. With this solution we have the following form for $(\hat{\delta}_i \hat{\phi}_j)$:

$$(\hat{\delta}_i \hat{\phi}_j) = \pm \hat{\phi} \delta_{ij} \mp \hat{\phi}_j \hat{\phi}_j / \hat{\phi} \tag{55}$$

To better see the nonlinear nature of this transformation, we can recall the definition of $(\hat{\delta}_i \hat{\phi}_j)$ and consider the finite integral of (55) for a given parameter ν_1 say. We have

$$\hat{\phi}_1' = (\hat{\phi}_1 \cosh \nu_1 \pm \hat{\phi} \sinh \nu_1)/Q, \qquad \hat{\phi}_k' = \hat{\phi}_k / Q, \qquad k = 2, 3 \tag{56}$$

where Q is given by

$$Q = (\hat{\phi} \cosh \nu_1 \pm \hat{\phi}_1 \sinh \nu_1)/\hat{\phi} \tag{57}$$

This particular realization differs topologically from those corresponding to the sterographic projections [or $O(3,1)/O(3)$ coset realizations (Hopkinson and Reya, 1972)]. In the above case the quantity $\hat{\phi}^2 = \hat{\phi}_i \hat{\phi}_i$ is invariant whereas in the latter the inner product summed over three dimensions is not invariant. The above particular realization corresponds to the $TC \times TC$ realization of Dalton (1979). The particular transformation pattern exhibited by it has been studied previously by several authors for other groups (Philips and Wigner, 1968; Melvin, 1962, 1963; Takabayasi, 1966). A detailed study has been given by Philips and Wigner (1968) for the de Sitter groups, a discussion in the context of the positive energy problem. To better understand the above-mentioned difference between these realizations and the sterographic projections (or coset realizations) the reader should contrast the realizations of Philips and Wigner with the conventional de Sitter realizations described in several classical studies (see for instance the study by Gürsey, 1964).

We now consider an explicit space-time functional form for the functions $\hat{\phi}_i$ which transform as described above. Let y_μ and x_μ represent coordinates of two points x and y in Minkowski space and define $r_\mu = x_\mu - y_\mu$. We try a solution in which the $\hat{\phi}_j$ are proportional to r_j,

$$\hat{\phi}_j = Z(r, r_4)r_j \tag{58}$$

where Z is to be determined.

Since we know that $(\hat{\delta}_i \hat{\phi}_j) = (\hat{\delta}_i Z)r_j + Z(\hat{\delta}_i r_j)$ and we know $(\hat{\delta}_i r_j) = -ir_4\delta_{ij}$, $(\hat{\delta}_i r_4) = ir_i$, we can use (58) and (55) and linear independence to obtain the following two conditions:

$$Z[ir_4 \pm r] = 0 \tag{59}$$

$$(\hat{\delta}_i Z) = Zr_i/r \tag{60}$$

From (59) we see that either $Z = 0$ or $ir_4 \pm r = 0$. If we choose to impose this explicit condition on the r_μ, we have $r_\mu r_\mu = 0$ and $r_4 = i\lambda r$, where in our notation we have $\lambda = +1$ for retarded and $\lambda = -1$ for advanced manifolds. We thus see that the two signs in (59) simply correspond to these two different manifolds. On these manifold we can use $ir_4 = \mp r$ to eliminate the explicit dependence of Z on r_4. Then using $(\hat{\delta}_i Z) = (\partial Z/\partial r)(\hat{\delta}_i r)$ in (60) we obtain the following differential equation:

$$\frac{\partial Z}{\partial r} = \frac{-Z}{r} \tag{61}$$

which has the simple solution

$$Z = q/r \tag{62}$$

where q is some constant. With (62) we have for $\hat{\phi}_i$ the solutions

$$\hat{\phi}_i = \frac{qr_i}{r}, \qquad r_\mu r_\mu = 0 \tag{63}$$

In the above we have considered an explicit solution of the $\hat{\phi}_i$ that did not have any direct dependence on the potentials A_μ, so that equations (28)–(31) were not involved. To obtain the Coulomb potential we now consider the transverse gauge $\partial_i A_i = 0$ and impose this as a Lorentz covariant gauge condition. Using (28)–(31) this covariance requirement (i.e., $\partial_i' A_i' = \partial_i A_i = 0$) is satisfied provided the following equations hold:

$$\nabla^2 \hat{\phi}_i = -i[\partial_4 A_i + \partial_i A_4] \tag{64}$$

$$\nabla^2 \phi_i = 0, \qquad \partial_i A_i = 0 \tag{65}$$

These equations are not relations which determine the $\hat{\phi}_i$ or A_μ, but rather, just conditions relating the two in the transverse gauge. From (64) and $\partial_i A_i = 0$ we can obtain the following relation for A_4:

$$\nabla^2(\partial_i \hat{\phi}_i + iA_4) = 0 \tag{66}$$

in the regions where the $\hat{\phi}_i$ and A_i are continuous (to obtain this relation one must interchange some partial derivatives). One solution of (66) is

$$A_4 = i\partial_i \hat{\phi}_i \tag{67}$$

If we use the condition $r_\mu r_\mu = 0$, we have the relation

$$\partial_\mu r_\rho = \delta_{\rho\mu} + \frac{\lambda \beta_\rho r_\mu}{r - \lambda \mathbf{r}\cdot\boldsymbol{\beta}} \tag{68}$$

where $r_4 = i\lambda r$, $\beta_4 = i$, and $\boldsymbol{\beta} = (1/c)\, d\mathrm{y}/dt$ represents the velocity at point y. With this relation we have

$$\partial_i(r_j/r) = \frac{\delta_{ij}}{r} + \frac{\lambda \beta_j r_i r - r_i r_j}{r^2(r - \lambda \mathbf{r}\cdot\boldsymbol{\beta})} \tag{69}$$

Using (67) with the solution in (63), we obtain the relation

$$A_4 = \frac{2iq}{r} \tag{70}$$

which is just the Coulomb potential in the transverse gauge. Since $\hat{\phi}_i$ must be imaginary (recall the conservation of probability) we notice from (63) that q must be imaginary. This leads to a real A_4 in (70). This is expected since we recall from Section 2 that we have absorbed the usual factor of i into the A_μ when we constructed the covariant derivative $D_\mu\psi = (\partial_\mu + A_\mu)\psi$. We emphasize that the relation (70) was derived for arbitrary velocities $\boldsymbol{\beta}$ of the point y. With the above result we see that the constant in the Coulomb potential represents the magnitude of the diagonal nonlinear pure Lorentz transformations. This gives an interesting picture of charge as the nonlinear counterpart of intrinsic spin in the sense that both are related to transformation generators of our space-time symmetry. We strongly emphasize, however, that the Coulomb potential follows from the particular realizations given above, and not from some of the other realizations discussed in Dalton (1979). In this picture, to say that a particle has an electric charge means that there exists a long-range distribution described by the $\hat{\phi}_i = qr_i/r$ (centered at the point y), which represent a particular nonlinear transformations of the wave function and potentials. As a last remark, the fact that the $\hat{\phi}_i$ do not fall off (i.e., $\hat{\phi}^2 = q^2$) with increasing r is a significant point. This is especially interesting in connection with the occurrence of the six ϕ_i together with equations (25)–(27) in a derivation of the classical Lorentz (Dalton, to be published) force directly from nonlinear realizations of the Lorentz group acting on the four-velocities of a particle.

7. OTHER SOLUTIONS

Here we briefly point out that the 30 equations discussed in Section 4 have other solutions that may be of physical interest. For this, we can use some previously published work. Let **S** represent a three-dimensional vector of Dalton (1979), and let **S*** represent its complex conjugate. If we write ϕ_i and $\hat{\phi}_i$ in the following way:

$$\phi_i = S_i - S_i^* \tag{71}$$

$$\hat{\phi}_i = -i(S_i + S_i^*) \tag{72}$$

we see that both ϕ_i and $\hat{\phi}_i$ will be pure imaginary. This condition is required by probability conservation. Now, one may show that with (71) and (72), most of the *nonlinear* realizations described in Dalton (1979) will satisfy the equations of Section 4. Among the variety of solutions described in Dalton

(1979), one finds a spinor (the $TE \times ZA$ realization). For it, we have

$$(\delta_i S_j) = \{\pm S_4 \delta_{ij} - \varepsilon_{ijk} S_k\}/2 \tag{73}$$

$$(\hat{\delta}_i S_j) = (-i)(\delta_i S_j) \tag{74}$$

where $S_4 = (D^2 - S^2)^{1/2}$ and D is some invariant.

This spinor is unusual in that the indices on the components S_k are the three Cartesian indices of space-time, and not indices of an internal space. The fact that **S** is a spinor follows from evaluating, from (73), the $O(3)$ Casimir invariant:

$$-(\delta_i(\delta_i S_k)) = \tfrac{3}{4} S_k \tag{75}$$

One may well ask how such a spinor could influence the interaction potential. Because of the nature of this spinor transformation it is difficult to obtain explicit solutions as in Section 6. One possibility is to consider an invariant Lagrangian for this spinor and to then solve the field equations to get an explicit space-time dependence. One possible invariant Lagrangian density for this realization is

$$\mathfrak{L}(S) = (\partial_\mu S_i)(\partial_\mu S_i) + \frac{(S_i \partial_\mu S_i)(S_j \partial_\mu S_j)}{D^2 - S^2} \tag{76}$$

Then, by choosing a specific gauge condition such as $A_4 = i\partial_i \hat{\phi}_i$ given in Section 6, one could calculate A_4. It should be clear then that the functional form for A_4 will probably differ from the $1/r$ Coulomb potential. The possibility that this spinor could be related to a neutrino, or to some other basic building block of matter, is an intriguing idea.

This example illustrates the potential richness of this approach to physical interactions. The fact that the $1/r$ Coulomb potential has been derived from one of these realizations strongly encourages investigation of these other nonlinear realizations. We should add that solutions other than those presented here and in Dalton (1979) have been found, and a paper describing them is in progress.

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APPENDIX A: DERIVATION OF COMMUTATOR RELATIONS

The commutator relations of a group of transformations are obtained by requiring that the transformations corresponding to the sequence $g(\alpha)g(\beta)g(\alpha^{-1})g(\beta^{-1})$ is itself an element of the transformation group, (i.e., closure). The commutator relations most often encountered are $[t_i, t_j] = C_{ijk}t_k$, where the t_k generate a matrix (linear) representation and the C_{ijk} are the group structure constants. This form, however, is not valid for more general transformations. Using our notation, we briefly outline the derivation for the commutator relations for an arbitrary transformation (linear or nonlinear), and from it obtain the form appropriate to the particular transformations studied in this paper. The transformations are defined on an arbitrary variable η, which could be either an operator (e.g., as occurs in quantum field theory), or a c-number function.

Under a given group element $g(\alpha)$ the variable η changes to η' as follows:

$$g(\alpha): \eta \to \eta' = F(g(\alpha), \eta) \tag{A.1}$$

where $F(g(0), \eta) = \eta$. We consider an infinitesimal transformation

$$\eta' = \eta + \alpha_i \frac{\partial F(g(\beta), \eta)}{\partial \beta_i}\bigg|_{\beta=0} + O(\alpha^2) \tag{A.2}$$

For convenience we use the notation $(\delta_i \eta)$ for the coefficient of α_i in (A.2). This function, often called a Killing vector, is defined as follows:

$$(\delta_i \eta) = \frac{\partial F(g(\beta), \eta)}{\partial \beta_i}\bigg|_{\beta=0} \tag{A.3}$$

The parenthesis used in this definition helps avoid confusion in composite expressions. If η can be written in the form $\eta = VW + X$ where V, W, and X are either operators, or c-number functions, then one can show from (A.2) the following result:

$$(\delta_i \eta) = (\delta_i V)W + V(\delta_i W) + (\delta_i X)$$

This property will be used in the following derivation. As a cautionary remark one should notice that if η is a function of noncommuting operators V_α, one generally cannot write $(\delta_i \eta) = (\partial \eta / \partial V_\alpha)(\delta_i V_\alpha)$. However, this procedure is valid if η is composed of c-number functions, or in special cases that do not involve an exchange of noncommuting operators.

For an arbitrary product $g(\alpha)g(\beta)$ we have the following relation through first order in α and β:

$$F(g(\alpha)g(\beta),\eta) = \eta + \beta_i(\delta_i\eta) + \alpha_i(\delta_i\eta) + \alpha_i\beta_j\big(\delta_i(\delta_j\eta)\big) \tag{A.4}$$

Using this expression in the transformation corresponding to $g(\alpha)g(\beta)$ $g(\alpha^{-1})g(\beta^{-1})$, and keeping only terms of order α and β gives the following result:

$$F\big(g(\alpha)g(\beta)g(\alpha^{-1})g(\beta^{-1}),\eta\big) = \eta + \alpha_i\beta_j\big[\big(\delta_i(\delta_j\eta)\big) - \big(\delta_j(\delta_i\eta)\big)\big] \tag{A.5}$$

If we impose closure, this transformation must be equivalent to $F(g(\gamma),\eta)$ where the parameter γ is a function of α and β. Using the expansion

$$F(g(\gamma),\eta) = \eta + \gamma_i(\delta_i\eta) \tag{A.6}$$

together with (A.5) we have

$$\gamma_i(\delta_i\eta) = \alpha_i\beta_j\big[\big(\delta_i(\delta_j\eta)\big) + \big(\delta_j(\delta_i\eta)\big)\big] \tag{A.7}$$

If we choose the solution

$$\gamma_k = \alpha_i\beta_j C_{ijk} \tag{A.8}$$

where C_{ijk} are constants (the structure constant) we have

$$\big(\delta_i(\delta_j\eta)\big) - \big(\delta_j(\delta_i\eta)\big) = C_{ijk}(\delta_k\eta) \tag{A.9}$$

The structure constants $C_{ijk} = -C_{jik}$ must also satisfy the Jacobi identity

$$C_{ijk}C_{kem} + C_{jek}C_{kim} + C_{eik}C_{kjm} = 0 \tag{A.10}$$

which follows from the associative property of the group multiplication. If C_{ijk} satisfies (A.10) then λC_{ijk} will also satisfy this equation. This freedom can be (and often is) used to choose different sets of parameters and/or functions $(\delta_i\eta)$ which satisfy (A.7)–(A.10). All such bases are equivalent, but often a particular one may be more convenient for a given problem.

For the particular applications in this paper, we are considering transformations of an N-dimensional column matrix ψ. We have

$$\psi'(x') = \psi(x) + \alpha_i(\delta_i\psi) \tag{A.11}$$

where

$$(\delta_i\psi) = -S_i\psi \tag{A.12}$$

Using this expression in (A.9) produces the following equation:

$$(\delta_i S_j) - (\delta_j S_i) + [S_i, S_j] = C_{ijk} S_k \tag{A.13}$$

Here $[S_i, S_j] \equiv S_i S_j - S_j S_i$ is the usual matrix commutator. If the S_i are inert under the group action [i.e., if $(\delta_i S_j) = 0$], equation (A.13) reduces to the ordinary matrix commutator relation for representations.

To solve (A.13) one can write it as a set of coupled nonlinear equations on the matrix elements of S_i. Another but related approach is to write S_i in the form

$$S_i = -g_{iC} M_C, \qquad 1 \leqslant C \leqslant N^2 \tag{A.14}$$

where the M_C are N^2 linearly independent matrices that are constant under the group action, i.e., $(\delta_i M_C) = 0$. Using (A.14) in (A.13) we arrive at the following relation:

$$(\delta_i g_{jE}) - (\delta_j g_{iE}) - g_{iC} g_{jD} V_{CDE} = C_{ijk} g_{kE} \tag{A.15}$$

where V_{CDE} is given by

$$[M_C, M_D] \equiv M_C M_D - M_D M_C = V_{CDE} M_E \tag{A.16}$$

We can write the matrix commutator $[M_C, M_D]$ as in (A.16) since the N^2 linearly independent matrices M_E form a basis for expansion of any $N \times N$ matrix. In most previous work, the solutions of (A.13) are limited by truncating the expansion in (A.14) to a subset of matrices which generate an N^2 matrix representation of the group. Equation (A.16) still holds for this subset. If these generator matrices are chosen such that $V_{ijk} = C_{ijk}$ we have the following relation:

$$(\delta_i g_{je}) - (\delta_j g_{ie}) - g_{ik} g_{jm} C_{kme} = C_{ijk} g_{ke} \tag{A.17}$$

where all indices in this expression range over the index set of group parameters. Solutions of (A.17) have been published for internal groups (Weinberg, 1968; Coleman et al., 1969; Callan et al., 1970; Joseph and Solomon, 1970; Salam and Strathdee, 1969; Julia and Luciani, 1980) as well as the Lorentz group (Dalton, 1980). In the most common special case of (A.17) the $g_{ij} = -\delta_{ij}$ for a given subgroup H of the group G. The remaining g_{ij} are functions of the coset parameters G/H.

For the work in this paper we consider the special case of (A.14) for which S_i has the form

$$S_i = -\phi_i 1 + t_i, \qquad [t_i, t_j] = C_{ijk} t_k \tag{A.18}$$

where the t_i are the constant $[(\delta_i t_j) = 0]$ generators of an N-dimensional representation, and 1 is the $N \times N$ unit matrix. Since $[1, t_j] = 0$ we have from (A.13) the relation

$$(\delta_i \phi_j) - (\delta_j \phi_i) = C_{ijk} \phi_k \tag{A.19}$$

In (A.19) we have a function ϕ_i for each parameter α_i. For the Lorentz group we are considering six functions ϕ_i. With (A.18), equation (A.11) takes on the following form:

$$\psi'(x') = \psi(x) + \alpha_i \phi_i 1 \psi(x) - \alpha_i t_i \psi(x) \tag{A.20}$$

so that with $\phi_i \neq 0$ we have simultaneous with the usual matrix transformation, a local (if $\partial_\mu \phi_i \neq 0$) diagonal realization of the group.

APPENDIX B: COVARIANT DERIVATIVE

To construct a covariant derivative appropriate to the transformations we are studying, we consider the following form:

$$D\Psi = \varepsilon_\mu (\partial_\mu + A_\mu) \Psi \tag{B.1}$$

Here ε_μ is a component of some four-vector for which the ε_μ are linearly independent of each other. This use of ε_μ is not necessary, but it makes it easier to include the four-vector transformation associated with the index μ. The transformation of A_μ in (B.1) is obtained as in local gauge theory from the condition that $D\Psi$ transform under the group like Ψ, that is

$$(D\Psi)' = UD\Psi \tag{B.2}$$

where $\Psi' = U\Psi$ and U is obtained from (A.17):

$$U = 1 + \alpha_i (\phi_i 1 - \hat{S}_i) \tag{B.3}$$

With $D' = \varepsilon'_\mu(\partial'_\mu + A'_\mu)$ we have from (B.2) the relation

$$\begin{aligned}\varepsilon'_\mu\partial'_\mu U\Psi + \varepsilon'_\mu A'_\mu U\Psi &= U\varepsilon_\mu\partial_\mu\Psi + U\varepsilon_\mu A_\mu\Psi \\ &= \varepsilon_\mu\partial_\mu U\Psi - \left[\varepsilon_\mu\partial_\mu, U\right]U^{-1}U\Psi + U\varepsilon_\mu A_\mu U^{-1}U\Psi \qquad \text{(B.4)}\end{aligned}$$

where the second line is a rearrangement of the right-hand side. Since $\varepsilon_\mu\partial_\mu$ is a Lorentz invariant, the first terms on each side cancel, giving

$$\begin{aligned}\varepsilon'_\mu A'_\mu U\Psi &= U\varepsilon_\mu A_\mu U^{-1}U\Psi - \left[\varepsilon_\mu\partial_\mu, U\right]U^{-1}U\Psi \\ \varepsilon'_\mu A'_\mu &= U\varepsilon_\mu A_\mu U^{-1} - (\varepsilon_\mu\partial_\mu U)U^{-1} \\ \varepsilon'_\mu A'_\mu &= \varepsilon_\mu\left[UA_\mu U^{-1} - (\partial_\mu U)U^{-1}\right] \qquad \text{(B.5)}\end{aligned}$$

From the four-vector property of ε_μ we have

$$\varepsilon_\nu = (\Lambda^{-1})_{\nu\mu}\varepsilon'_\mu \qquad \text{(B.6)}$$

where the $\Lambda_{\mu\nu}$ are elements of a four-vector transformation. Using this in (B.5) and the linear independence of the ε'_μ, we obtain the relation

$$A'_\mu = \Lambda^{-1}_{\nu\mu}\left[UA_\nu U^{-1} - (\partial_\nu U)U^{-1}\right] \qquad \text{(B.7)}$$

Apart from the factor of $\Lambda^{-1}_{\nu\mu}$, equation (B.7) has the same form obtained for transformations of Yang–Mills gauge potentials. There is, however, an important difference! In nonlinear realizations, the space-time dependence of U arises through the generators (here via the ϕ_i, or g_{ij} functions), whereas in gauge theory it enters through the group parameters. Since the functions ϕ_i and g_{ij} entering the generators are not inert under the group action, they cannot be incorporated with the parameters to reduce the nonlinear realizations to a gauge transformation (an exception to this is the case where the group is Abelian).

Using the infinitesimal form (B.3), equation (B.7) reduces to the equation

$$A'_\mu = A_\mu - \alpha_i\left[s_i^{\mu\nu}A_\nu + \partial_\mu\phi_i\right] \qquad \text{(B.8)}$$

Here the $s_i^{\mu\nu}$ are the elements of the infinitesimal four-vector representation generators in a given basis;

$$\Lambda_{\mu\nu} = \delta_{\mu\nu} - \alpha_i s_i^{\mu\nu} \qquad \text{(B.9)}$$

If we *define* the quantity $(\delta_i A_\mu)$ as follows:

$$(\delta_i A_\mu) \equiv \left.\frac{\partial A'}{\partial \alpha_i}\right|_{\alpha=0} \tag{B.10}$$

and make a Taylor's expansion for A'_μ on the parameter α_i in (B.8) we obtain the relation

$$(\delta_i A_\mu) = -s_i^{\mu\nu} A_\nu - \partial_\mu \phi_i \tag{B.11}$$

From (B.8) or (B.11) we see that the diagonal realizations cause a modification of the usual four-vector transformation properties of A_μ by the presence of a derivative term $\partial_\mu \phi_i$. This term is important in two ways. First, only solutions for A_μ which transform as in (B.8) will be allowed. Second, the term $\partial_\mu \phi_i$ will contribute to the conserved total angular momentum (see Section 3 for discussion).

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